The Dynamic Distribution in the Fixed Cost Model: An Analytical Solution

Summer Workshop on Money, Banking, Payments and Finance
Bank of Canada

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Motivation

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- ... but in general, analytical solution is hard because the PDE is *endogenous*: evolution depends on the flow of resets
- Existing methods require shortcuts (e.g. symmetry, small shocks)

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  - Size-dependence: large enough shocks can even reverse effect of a monetary shock
  - State-dependence: shocks to steady state are not general
  - Trend inflation is easy to handle, not simply "second-order"

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- Fixed costs imply an inaction region $x \in [a < 0 < b]$
- Outside the inaction region, pay a fixed cost and reset to $x = 0$
- Macroeconomic outcomes depend on the distribution $h(x, t)$

What Characterizes the Distribution $h(x, t)$?

1. $h(x, t)$ satisfies the Kolmogorov Forward Equation on interval $[a, 0) \cup (0, b]$:

$$\partial_t h(x, t) = \gamma \partial_x^2 h(x, t) - \mu \partial_x h(x, t) - \eta h(x, t) \tag{1}$$

   - γ: 2x variance of Brownian motion
   - μ: drift in x
   - η: random reset rate

2. ... subject to constraints:
   - *Continuity condition*: $h(x, t)$ continuous at $x = 0$
   - *Dirchlet boundary conditions* (a and b are absorbing barriers):

$$h(a, t) = 0 \qquad h(b, t) = 0$$

   - *Probability conservation*: $\int_a^b h(x, t) dx = 1$ for all t
   - *Initial condition*: $h(x, 0) = \phi(x)$

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$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) - \mu \partial_x h(x,t) - \eta h(x,t) + \underbrace{\delta(x)\left(F(t) + \eta\right)}_{\text{Endog. component}}$$

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- This would be easy if we knew the reset frequency $F(t)$!

How does the Reset Frequency Depend on the Distribution?

- **Lemma:** Endogenous flow of probability out of $[a, b]$:

$$F(t) = (\gamma \partial_x h(a,t) - \mu h(a,t)) - (\gamma \partial_x h(b,t) - \mu h(b,t))$$

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- Endogenous: $F(t)$ depends on $h(x,t)$, which depends on $F(t)$...
- Start by finding the conditional solution for $h(x,t)$

How does the Distribution Depend on the Reset Frequency?

- The textbook (conditional) solution to the KFE is

$$h(x,t) = \int_a^b G(x,y,t)\phi(y)dy + \int_0^t G(x,0,t-\tau)\left(F(\tau)+\eta\right)d\tau$$

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- which is convenient when written in terms of the Green's function

$$G(x,y,t) = \sum_{n=1}^{\infty} X_n(x)X_n(-y)e^{-\lambda_n t}$$

$$X_n(x) \equiv \sqrt{\frac{2}{b-a}} e^{\frac{\mu}{2\gamma}(x-a)} \sin\left(\frac{n\pi(x-a)}{(b-a)}\right) \qquad \lambda_n \equiv \frac{\gamma n^2\pi^2}{(b-a)^2} + \frac{\mu^2}{4\gamma} + \eta$$

Everything is Better in Frequency Space

- Messy PDE $\implies$ try the Laplace transform, e.g.

$$\hat{F}(s) \equiv \mathcal{L}\{F\}(s) = \int_0^\infty F(t)e^{-st}dt$$

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- Laplace transforms of KFE solution and frequency equation are:

$$\hat{h}(x,s) = \int_a^b \hat{G}(x,y,s)\phi(y)dy + \hat{G}(x,0,s)\left(\hat{F}(s) + \eta\right)$$

$$\hat{F}(s) = \left(\gamma\partial_x\hat{h}(a,s) - \mu\hat{h}(a,s)\right) - \left(\gamma\partial_x\hat{h}(b,s) - \mu\hat{h}(b,s)\right)$$

We Can Solve the Reset Frequency without the Distribution

Lemma
The reset frequency satisfies

$$\hat{F}(s) = \frac{\alpha(s)}{1 - \beta(s)}$$

where

$$\beta(s) = \sum_{n=1}^{\infty} \frac{\beta_n}{s + \lambda_n} \qquad \alpha(s) = \sum_{n=1}^{\infty} \frac{\alpha_n}{s + \lambda_n}$$

$$\theta_n \equiv \left(\gamma \left(X_n'(a) - X_n'(b)\right) - \mu \left(X_n(a) - X_n(b)\right)\right)$$

$$\beta_n \equiv \theta_n X_n(0) \qquad \alpha_n \equiv \theta_n \int_a^b X_n(-y)\phi(y)dy + \eta\beta_n$$

▸ Proof

Invert the Transform to Get the Distribution

Theorem

The distribution $h(x,t)$ is given by

$$h(x,t) = \mathcal{L}^{-1}\{\hat{h}\}(x,s)$$

where

$$\hat{h}(x,s) = \int_a^b \hat{G}(x,y,s)\phi(y)dy + \hat{G}(x,0,s)\left(\frac{\alpha(s)}{1-\beta(s)} + \eta\right)$$

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- Specifically: transformed $\hat{Z}(s)$ is *linear* in $\hat{F}(s)$ in the frequency space

Transformed, All Macro Dynamics are Linear in Reset Frequency

Lemma

The transformed aggregate $\hat{Z}(s)$ satisfies

$$\hat{Z}(s) = \alpha^Z(s) + \beta^Z(s)\hat{F}(s)$$

where

$$\beta^Z(s) \equiv \int_a^b f_Z(x)\hat{G}(x, 0, s)dx$$

$$\alpha^Z(s) \equiv \int_a^b f_Z(x) \int_a^b \hat{G}(x, y, s)\phi(y)dydx + \eta\beta^Z(s)$$

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- We can solve how the distribution evolves from the initial condition without resets (easy)
- What's left? How the distribution responds to new agents entering at $x = 0$ at rate $F(t)$.
- $Z(t)$ depends only on $h(x, t)$ which depends only on $F(t)$. Skip the intermediate step!

In Most Cases This is Easy

- Could get hairy if $f_Z(x)e^{\lambda x}$ doesn't integrate nicely (rare)
- In paper, derive expressions for $\alpha^Z(s)$, $\beta^Z(s)$ for common integrating functions:
  - $f_Z(x) = e^{\psi x}$: exponential, e.g. Golosov and Lucas (2007)
  - $f_Z(x) = x$: average state, e.g. Alvarez et al (2024)
  - $f_Z(x) = x^2$: second moment

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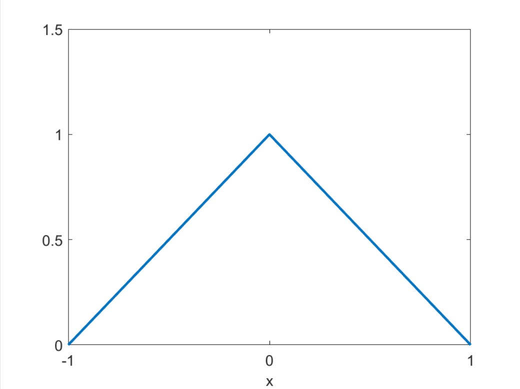
- No drift, symmetric menu costs $\implies$ inaction region is $[-b, b]$
- Aggregate output is determined by

$$\underbrace{Y(t)^{\eta(\epsilon-1)}\alpha^{\epsilon-1}e^{(\epsilon-1)\mu^*}}_{Z(t)} = \int_a^b \underbrace{e^{(1-\epsilon)x}}_{f_Z(x)} h(x,t)dx \tag{3}$$

Effects of a Monetary Shock: Initial Condition



(a) Stationary distribution $h(x, \infty)$

(b)

Figure 1: Money Supply Increase Reduces Markup Gaps

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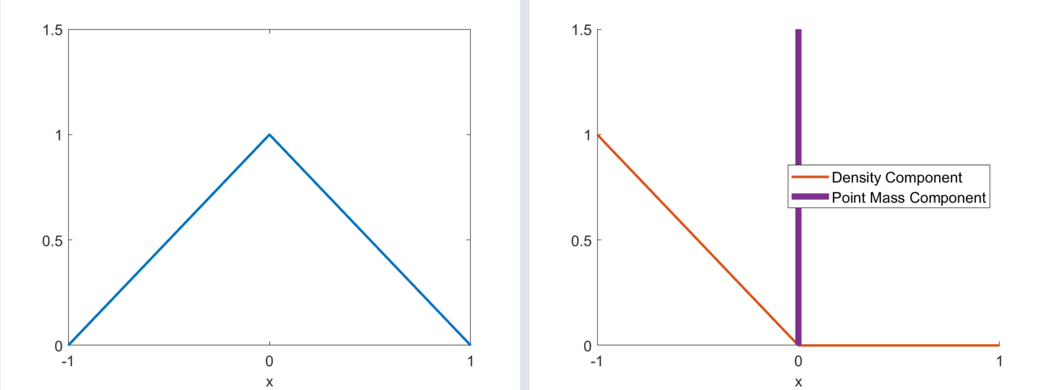

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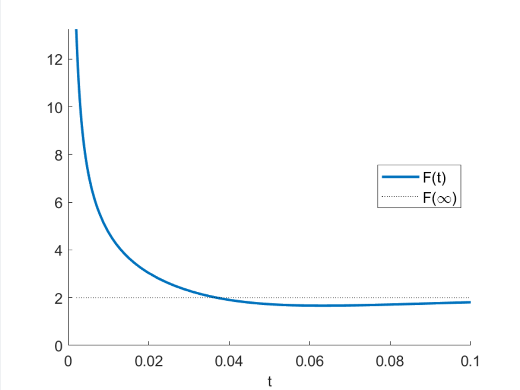

(a) Frequency of price resets

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Figure 2: Markup Gaps Return to Stationary Distribution

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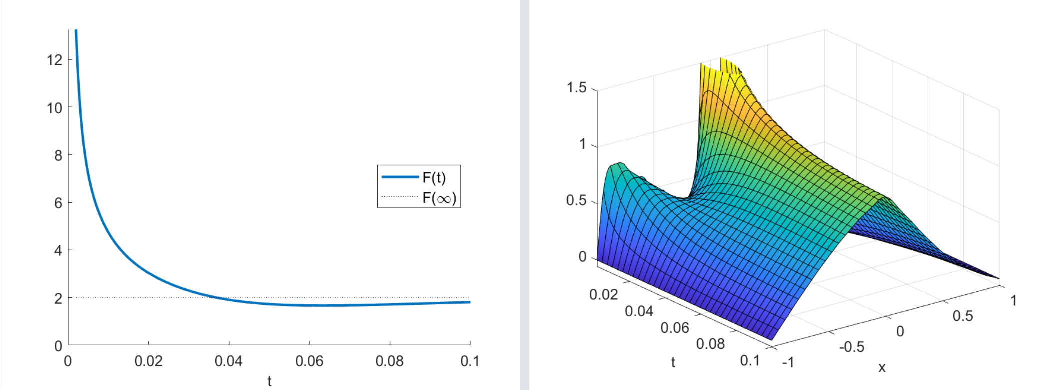

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$$\phi_{\Delta}(x) = \bar{h}(x + \Delta) + \delta(x) \int_{a}^{a+\Delta} \bar{h}(x) dx$$

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- Summarize with the output IRF:

$$IRF^Y(t) = \log Y(t) - \log Y(\infty) \tag{4}$$

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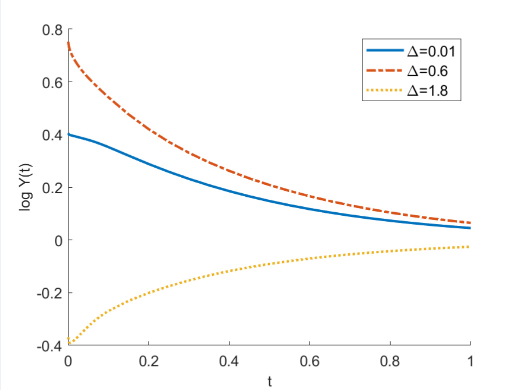

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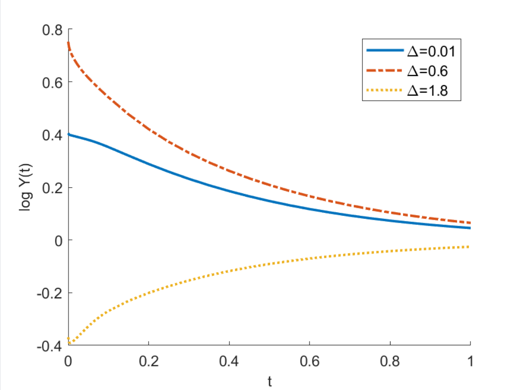

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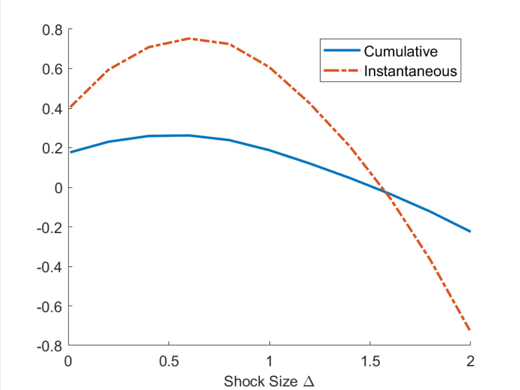

(b) Instantaneous Effects and CIRs

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  - Extreme case: all firms reset = mean-preserving reduction in variance. Jensen's inequality implies average $e^{(1-\epsilon)x}$ falls.

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  - Followed by small $\Delta = 0.01$ shock time t
- Delay t matters, even though shock is small: shocks to stationary distribution have the largest effects!

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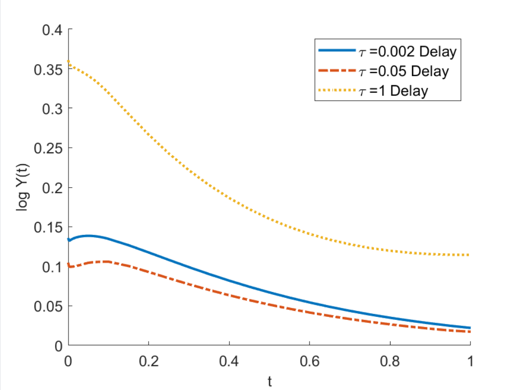

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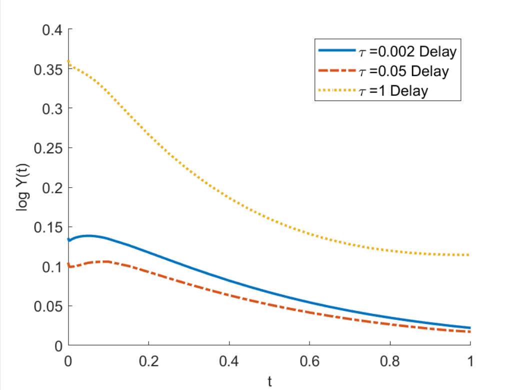

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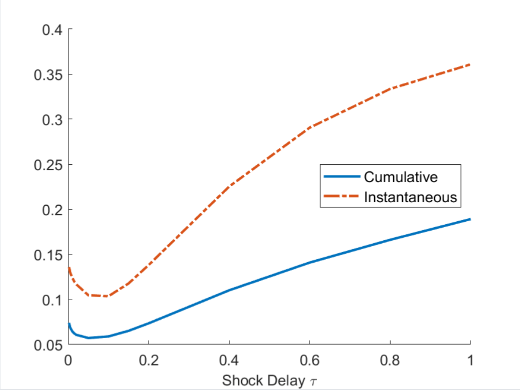

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  - Nonlinearities (e.g. size-dependence, history-dependence)

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  - Ongoing work: analytical solution with optimal boundaries and small shocks
- The reset frequency is key! (determines distribution, IRFs)
- Advances our tools to understand:
  - IRFs, especially nonlinear aggregation/higher moments
  - Nonlinearities (e.g. size-dependence, history-dependence)
  - Asymmetries (e.g. trend inflation)

Conclusions

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  - More?

First define $\beta(s) \equiv (\gamma\partial_x - \mu)\left(\hat{G}(a,0,s) - \hat{G}(b,0,s)\right)$ and $\alpha(s) \equiv \int_a^b (\gamma\partial_x - \mu)\left(\hat{G}(a,y,s) - \hat{G}(b,y,s)\right)\phi(y)dy + \eta\beta(s)$:

$$(\gamma\partial_x - \mu)\,\hat{h}(x,s) = \int_a^b (\gamma\partial_x - \mu)\,\hat{G}(x,y,s)\phi(y)dy + (\gamma\partial_x - \mu)\,\hat{G}(x,0,s)\left(\hat{F}(s) + \eta\right)$$

$$\implies (\gamma\partial_x - \mu)\left(\hat{h}(a,s) - \hat{h}(b,s)\right) = \alpha(s) + \beta(s)\hat{F}(s)$$

$$\implies \hat{F}(s) = \frac{\alpha(s)}{1 - \beta(s)}$$

Then derive expressions for $\alpha(s)$ and $\beta(s)$.

$$\beta(s) = \left(\gamma\partial_x \hat{G}(a,0,s) - \mu\hat{G}(a,0,s)\right) - \left(\gamma\partial_x \hat{G}(b,0,s) - \mu\hat{G}(b,0,s)\right)$$

$$= \sum_{n=1}^{\infty} (\gamma\partial_x - \mu)\left(X_n(a) - X_n(b)\right) X_n(0)\, \hat{T}_n(s) = \sum_{n=1}^{\infty} \beta_n \hat{T}_n(s) = \sum_{n=1}^{\infty} \frac{\beta_n}{s+\lambda_n}$$

$$\alpha(s) = \int_a^b (\gamma\partial_x - \mu)\left(\hat{G}(a,y,s) - \hat{G}(b,y,s)\right)\phi(y)dy + \eta\beta(s)$$

$$= \sum_{n=1}^{\infty} (\gamma\partial_x - \mu)\left(X_n(a) - X_n(b)\right)\left(\int_a^b X_n(-y)\phi(y)dy\right) \hat{T}_n(s) + \eta\beta(s)$$

$$= \sum_{n=1}^{\infty} (\gamma\partial_x - \mu)\left(X_n(a) - X_n(b)\right)\left(\int_a^b X_n(-y)\phi(y)dy\right) \frac{1}{s+\lambda_n} + \eta\frac{\beta_n}{s+\lambda_n}$$

$$= \sum_{n=1}^{\infty} \frac{\alpha_n}{s+\lambda_n}$$

▸ Back

$$
\begin{aligned}
\hat{Z}(s) &= \int_a^b f_Z(x)\hat{h}(x,s)dx \\
&= \int_a^b f_Z(x)\left(\int_a^b \hat{G}(x,y,s)\phi(y)dy + f_Z(x)\hat{G}(x,0,s)\left(\hat{F}(s)+\eta\right)\right)dx \\
&= \int_a^b f_Z(x)\left(\int_a^b \hat{G}(x,y,s)\phi(y)dy + \hat{G}(x,0,s)\eta\right)dx + \int_a^b f_Z(x)\hat{G}(x,0,s)dx\hat{F}(s) \\
&= \alpha^Z(s) + \beta^Z(s)\hat{F}(s)
\end{aligned}
$$